

On local equivalence problem of spacetimes with two orthogonally transitive commuting Killing fields

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Abstract

Considered is the problem of local equivalence of generic four-dimensional metrics possessing two commuting and orthogonally transitive Killing vector fields. A sufficient set of eight differential invariants is explicitly constructed, among them four of first order and four of second order in terms of metric coefficients. In vacuum case the four first-order invariants suffice to distinguish generic metrics.

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1 Introduction

The metric problem of equivalence is the problem to decide whether two pseudo-Riemannian spaces are locally isometric. This question is of importance in general relativity ([39, Ch. 9]), where isometric means gravitationally identical.

Existing classifying algorithms depend on computation of curvature invariants, which are components of Riemann curvature tensor and its covariant derivatives in a particular frame specified in the course of the algorithm. This is the central idea of Cartan's [8, 6] solution to the equivalence problem by the method of moving frames, and the Karlhede [22, 24] algorithm derived from it. Karlhede classification lies behind two projects of publicly available databases of exact solutions of Einstein equations [21, 36]. In vacuum case, the Riemann tensor can be replaced with its traceless part, the Weyl tensor. Presently, the Cartan–Karlhede curvature invariants are the only invariants known to solve the metric equivalence problem in full generality.

The main difficulty connected with the Cartan–Karlhede algorithm lies with the size of the data generated at the n th iteration of the main loop. The number and complexity of generated curvature invariants grow quite rapidly with the order n of covariant derivatives. During the past two decades, much effort has been exerted to tight the bounds on n , with recent limit values being collected in [39, Ch. 9, Table 9.1]. The required n can be as high as five, even though the upper limits are reached only in very special cases [38]. Understandably, the higher n is, the bigger portion of generated invariants are actually redundant even if algebraic identities are accounted for as in MacCallum and Åman [30]. This is the price paid for leaving no metrics aside. From the computational perspective, one is naturally interested in solving the equivalence problem in terms of minimal possible number of invariants of lowest possible order. In this paper we provide such a solution within the class \mathcal{T} of

generic metrics that possess two commuting orthogonally transitive Killing vectors. Our choice reflects the abundance of exact solutions of class \mathfrak{T} in general relativity, be it stationary axisymmetric [10] or cylindrically symmetric spacetimes. The orthogonal transitivity is an additional simplification that not only occurs for all explicitly known stationary axisymmetric metrics [39, p. 294], but can be derived from much simpler property of invertibility [35, 9]. A fairly large portion of explicitly known spacetime metrics belong to this class, mainly due to the fact that powerful generation techniques operate within \mathfrak{T} or its vacuum subclass. Reduction to a rather compact Ernst equation [17] was the key to solution by the inverse scattering method of Belinsky and Zakharov [4], as well as a number of Bäcklund transformations, such as the Harrison transformation [19], Hoenselaers–Kinnersley–Xanthopoulos transformation [20], Kinnersley–Chitre transformation [28], and Neugebauer transformation [31]. The majority of these results have been extended to metrics with suitable energy-momentum tensors, notably electro-magnetism. For relations between different solution-generating techniques see Cosgrove [14]. For a number of explicit solutions see [39, 40]. As shown by Kinnersley [27], every vacuum Petrov type D metric belongs to \mathfrak{T} , the result having been extended to aligned electro-vacuum metrics with cosmological constant by Debever and McLenaghan [15]. On the other hand, metrics from the class \mathfrak{T} can be of any Petrov type except III . Last but not least, Cosgrove [12] almost solved the equivalence problem for vacuum metrics in this class. It should be noted that metrics of class \mathfrak{T} pass the Karlhede classification rather smoothly, requiring no more than two covariant derivatives to be taken. The Riemann tensor itself being of order two, the computed invariants are of order up to four in terms of metric coefficients. It may therefore come as a surprise that appropriately chosen invariants of first and second order suffice, at least for generic metrics. Our goal is to demonstrate that.

The content of the paper is as follows. In Section 2 we review the transformations that preserve the class \mathfrak{T} . Then we compute the number of lowest order invariants to show that four of them are of the first order.

In Section 3 we present the four independent invariants $C_\rho, C_\chi, Q_\chi, Q_\gamma$ as well as an invariant frame X, Y on J^1 . They provide us with a solution to the equivalence problem rather immediately. It should be stressed that we consider only generic metrics within our class. The exact meaning of “generic” is specified along the solution of the equivalence problem. In particular, metrics are required to satisfy the condition that their Killing algebra possesses a unique two-dimensional commutative subalgebra. However, we leave open the problem of recognizing this condition in terms of first-order invariants.

Our work was to a great extent inspired by Cosgrove’s treatment [12] of the subclass of vacuum space-times. Exploiting an auxiliary invariantly defined metric of constant negative curvature, Cosgrove [12, 13] devised a method to find new vacuum solutions of Einstein equations without cosmological constant. The work [12] solves the equivalence problem for empty spacetimes modulo transformations of the natural harmonic coordinate r and its companion coordinate.

In Section 4 we attempt a comparison between our and Petrov/Karlhede classification. The Cosgrove invariant Q_γ is rather immediately identified with imaginary part of Ψ_2 , another, C_χ , with the real part of Ψ_2 in case of Ricci-flat (i.e., vacuum) metrics. We were unable to identify the others. The last section is devoted to examples and discussion.

Beyond the scope of this paper we leave investigation of the non-generic cases as well as extension to wider classes of metrics. A general classification of conditions that

can be put on a two-dimensional Killing distribution and its orthogonal complement can be found in [11, 3] along with the description of general solution in nongeneric cases.

2 The metric and the pseudogroup

2.1 The metric

In this section we introduce the class \mathfrak{T} of pseudo-Riemannian metrics of interest in this paper and the pseudogroup of local diffeomorphisms they admit. Let \mathcal{M} be a four-dimensional manifold. We consider a non-degenerate metric \mathbf{g} on \mathcal{M} possessing a unique two-dimensional commutative algebra \mathfrak{K} of Killing vectors. Let Ξ denote the vector distribution on \mathcal{M} generated by \mathfrak{K} . Then Ξ is two-dimensional, since no non-constant multiple of a non-zero Killing vector can be a Killing vector. Let ξ_1, ξ_2 denote arbitrary generators of the algebra \mathfrak{K} . We assume additionally that the Killing vectors be *orthogonally transitive* [35, 9], meaning that the distribution Ξ^\perp generated by the covectors $\mathbf{g}(\xi_i, \cdot)$ is Frobenius integrable and transversal to Ξ . Hence the tangent spaces decompose as $T_a\mathcal{M} = \Xi_a \oplus \Xi_a^\perp$. By Frobenius integrability, the manifold becomes locally foliated by a two-parameter family \mathcal{S} of two-dimensional leaves of the distribution Ξ^\perp . For an arbitrarily chosen leaf \mathcal{S}_0 , points of \mathcal{S}_0 parametrize the orbit space \mathcal{M}/Ξ , at least locally. Note that global properties of the foliation are irrelevant to the local equivalence problem.

Metrics of class \mathfrak{T} have a simple description in terms of *adapted coordinates* (see also Weyl–Lewis–Papapetrou coordinates below). Denoting by \mathcal{S}_0 a fixed surface of the family \mathcal{S} , we arbitrarily choose coordinates t^1, t^2 in an open subset of \mathcal{S}_0 and extend them along the trajectories of the fields ξ_k to functions t^1, t^2 in an open subset U of \mathcal{M} . Shrinking U if necessary, we determine functions u^1, u^2 in U by the requirement that $\xi_l u^k = \delta_l^k$ and $\mathcal{S}_0 \cap U = \{u^1 = u^2 = 0\}$. In the coordinate system (t^1, t^2, u^1, u^2) we have $\xi_k = \partial/\partial u^k$, $k = 1, 2$, while the metric \mathbf{g} assumes the form

$$\mathbf{g} = g_{ij} dt^i dt^j + h_{kl} du^k du^l, \quad (1)$$

where $g_{ij} = g_{ji}$, $h_{kl} = h_{lk}$ are functions on \mathcal{S} , i.e., depend on t^1, t^2 only. The induced metric $\mathbf{g}|_{\mathcal{S}}$ on \mathcal{S} is $g = g_{ij} dt^i dt^j$. Obviously,

$$\det \mathbf{g} = \det g \det h \neq 0.$$

The other leaves of the foliation \mathcal{S} are $u^1 = \text{const}_1$, $u^2 = \text{const}_2$. Since shifts along the trajectories of Killing vectors are isometries, the particular choice of the leaf \mathcal{S}_0 is irrelevant. Henceforth we drop the index 0 and write simply \mathcal{S} instead of \mathcal{S}_0 .

The metric \mathbf{g} can be identified with a section of the second symmetric power $S^2 T^* \mathcal{M} \rightarrow \mathcal{M}$ of the cotangent bundle $T^* \mathcal{M} \rightarrow \mathcal{M}$. The two-dimensional spaces Ξ_a , $a \in \mathcal{S}$, constitute a vector bundle $\Xi \rightarrow \mathcal{S}$. Considering the decompositions $T_a \mathcal{M} = \Xi_a \times T_a \mathcal{S}$ for $a \in \mathcal{S}$, the metric $g = \mathbf{g}|_{\mathcal{S}}$ can be identified with a section of $S^2 T^* \mathcal{S} \rightarrow \mathcal{S}$, while the remaining metric coefficients $h_{kl} = \mathbf{g}(\xi_k, \xi_l)$ parametrize sections of the bundle $S^2 \Xi^* \rightarrow \mathcal{S}$. Thus, locally, metrics \mathbf{g} are in a one-to-one correspondence with sections (g, h) of the six-dimensional vector bundle $\pi : S^2 T^* \mathcal{S} \times S^2 \Xi^* \rightarrow \mathcal{S}$. This correspondence, given simply by (1) in adapted coordinates, is only valid under the condition that the distributions Ξ and Ξ^\perp are fixed.

2.2 The pseudogroup

Turning back to our classification problem, we need to know the corresponding local transformation pseudogroup acting on sections of the bundle π . Otherwise said, what are local diffeomorphisms Φ of \mathcal{M} such that $\mathbf{g}' = \Phi^*\mathbf{g}$ is of the same symmetry properties as \mathbf{g} (both have the same algebra \mathfrak{K} , hence the same generators ξ_1, ξ_2 in \mathfrak{K} , and induce one and the same decomposition $T_a\mathcal{M} = \Xi_a \oplus \Xi_a^\perp$)? More to the point, what they are in terms of g and h ?

Let \mathfrak{G} denote the pseudogroup of local diffeomorphisms of the base \mathcal{S} of π . These have a unique lift along the bundle $S^2T^*\mathcal{S} \rightarrow \mathcal{S}$, given by $g' = \phi^*g$, $\phi \in \mathfrak{G}$. Next consider the standard representation of GL_2 in the two-dimensional vector space of the algebra \mathfrak{K} . That is, given $(a_l^k) \in \mathrm{GL}_2$, let $\xi'_k = a_k^i \xi_i$. Then we have the induced representation in each Ξ_a , hence in each $S^2\Xi_a^*$, namely $h'_{kl} = \mathbf{g}(\xi'_k, \xi'_l) = a_k^i a_l^j h_{ij}$. Thus we arrive at the product $\mathfrak{G} \times \mathrm{GL}_2$, henceforth referred to as the pseudogroup of transformations of the total space $S^2\Xi^* \times S^2T^*\mathcal{S}$ of π .

Proposition 1 *Let two metrics \mathbf{g} and \mathbf{g}' on \mathcal{M} possess a unique two-dimensional commutative algebra \mathfrak{K} of Killing vectors, which induces one and the same decomposition $T_a\mathcal{M} = \Xi_a \oplus \Xi_a^\perp$, $a \in \mathcal{M}$. Then $\mathbf{g}' = \Phi^*\mathbf{g}$ locally with respect to a diffeomorphism Φ of \mathcal{M} if and only if the corresponding sections (g, h) and (g', h') of π are locally equivalent with respect to the action of $\mathfrak{G} \times \mathrm{GL}_2$ given above.*

Proof. To prove the ‘if’ part, let ϕ denote a local diffeomorphism on \mathcal{S} such that $g' = \phi^*g$ and let $(a_j^i) \in \mathrm{GL}_2$ be such that $h'_{kl} = a_k^i a_l^j h_{ij}$. Denoting by Φ the local diffeomorphism of \mathcal{M} with coordinate description $u^k \circ \Phi = a_l^k u^l$, $t^i \circ \Phi = t^i \circ \phi$, from (1) we get $\Phi^*\mathbf{g}' = \mathbf{g}$.

To prove the converse, consider a local diffeomorphism Φ of \mathcal{M} such that $\mathbf{g}' = \Phi^*\mathbf{g}$. Let ξ_1, ξ_2 be the basis of \mathfrak{K} . Then $\Phi_*\xi_1, \Phi_*\xi_2$ are Killing vectors for \mathbf{g} and belong to the unique commutative subalgebra \mathfrak{K} , therefore they are linear combinations of ξ_1, ξ_2 with constant coefficients. Denoting by α the local diffeomorphism of \mathcal{M} with local description $u^k \circ \alpha = a_l^k u^l$, $(a_k^l) \in \mathrm{GL}_2$, $t^i \circ \alpha = t^i$, we have $\alpha_*\xi_k = \Phi_*\xi_k$. Hence, $\Phi \circ \alpha^{-1}$ preserves both vectors ξ_1, ξ_2 . The coefficients $(a_k^l) \in \mathrm{GL}_2$ being uniquely determined, α can be regarded as a representation of GL_2 by local diffeomorphisms of \mathcal{M} , equivalent to the standard representation of GL_2 in each Ξ_a .

Now, $\Phi\alpha^{-1}\mathcal{S} = \Phi\mathcal{S}$ is another leaf of the family, i.e., $\beta\mathcal{S}$, where β is a shift along the trajectories of ξ_1, ξ_2 , i.e., $u^l \circ \beta = u^l + b^l$. Denoting $\phi := \beta^{-1} \circ \Phi \circ \alpha^{-1}$, we see that ϕ is a local diffeomorphism that preserves \mathcal{S} as well as both vector fields ξ_1, ξ_2 . Moreover, ϕ is uniquely determined by its restriction $\phi|_{\mathcal{S}}$, hence ϕ effectively belongs to the pseudogroup \mathfrak{G} . Thus the local diffeomorphism Φ can be decomposed as $\Phi = \beta \circ \phi \circ \alpha$ into an automorphism $\alpha \in \mathrm{GL}_2$, a local diffeomorphism of $\phi \in \mathfrak{G}$, and a shift. However, shifts are isometries, hence $\mathbf{g}' = \Phi^*\mathbf{g} = \alpha^*\phi^*\beta^*\mathbf{g} = \alpha^*\phi^*\mathbf{g}$. Thus we are left with the pseudogroup $\mathfrak{G} \times \mathrm{GL}_2$. \square

The classification problem for metrics \mathbf{g} thus reduces to identifying orbits of the action of $\mathfrak{G} \times \mathrm{GL}_2$ on sections of the bundle π . Following Lie’s classical method, one has to find a sufficient number of independent scalar differential invariants of the action. These are functions on the jet prolongation $J^\infty\pi$ that are invariant with respect to the infinitesimal action of $\mathfrak{G} \times \mathrm{GL}_2$ on $J^\infty\pi$. Thus the first steps to be done are to describe the action of the corresponding Lie algebra $\mathfrak{g} \times \mathfrak{gl}_2$ on π and its extension to the jet prolongation $J^\infty\pi$.

Starting from the description of elements of \mathfrak{G} as diffeomorphisms ϕ given by $t^k \circ \phi = \phi^k(t^1, t^2)$, we see that \mathfrak{g} acts by vector fields

$$U_\psi = \psi^i \frac{\partial}{\partial t^i} \quad (2)$$

with $\psi = (\psi^1(t^1, t^2), \psi^2(t^1, t^2))$ being an arbitrary couple of functions on \mathcal{S} . To determine the extension U_ψ^π of infinitesimal transformations (2) on π , we require that the Lie derivative $U_\psi^\pi(g) = U_\psi^\pi(g_{ij} dt^i dt^j)$ be zero, i.e.,

$$U_\psi^\pi g_{ij} = -g_{is} \frac{\partial \psi^s}{\partial t^j} - g_{sj} \frac{\partial \psi^s}{\partial t^i}.$$

Thus we have

$$U_\psi^\pi = \psi^i \frac{\partial}{\partial t^i} - \sum_{i \leq j} \left(g_{is} \frac{\partial \psi^s}{\partial t^j} + g_{sj} \frac{\partial \psi^s}{\partial t^i} \right) \frac{\partial}{\partial g_{ij}}.$$

The remaining infinitesimal generators are V_a with $a \in \mathfrak{gl}_2$ corresponding to action of GL_2 on components h_{kl} . In terms of coordinates u^k on \mathcal{M} we have $V_a = a_l^k u^l (\partial / \partial u^k)$ with $a = (a_l^k) \in \mathfrak{gl}_2$ being an arbitrary constant 2×2 matrix, hence

$$V_a^\pi h_{kl} = -h_{ks} a_l^s - h_{sl} a_k^s$$

similarly as above. Then

$$V_a^\pi = - \sum_{k \leq l} (h_{ks} a_l^s + h_{sl} a_k^s) \frac{\partial}{\partial h_{kl}}, \quad (3)$$

since components transversal to \mathcal{S} are suppressed on π .

Next we consider the infinite-dimensional vector bundle $J^\infty \pi$. Recall that obvious coordinates along the fibres of $J^\infty \pi$ are the formal derivatives of g_{ij} and h_{kl} of all orders with respect to t^1, t^2 , e.g., $g_{ij,k} = \partial g_{ij} / \partial t^k$, etc. On $J^\infty \pi$ one has the usual total derivative

$$D_k = \frac{\partial}{\partial t^k} + g_{ij,k} \frac{\partial}{\partial g_{ij}} + h_{ij,k} \frac{\partial}{\partial h_{ij}} + g_{ij,kl} \frac{\partial}{\partial g_{ij,l}} + h_{ij,kl} \frac{\partial}{\partial h_{ij,l}} + \dots$$

By $\bar{U}_\psi := U_\psi^{J^\infty \pi}$ and $\bar{V}_a := V_a^{J^\infty \pi}$ we denote the well-known prolonged fields on $J^\infty \pi$, characterized by the defining relation $[\bar{U}_\psi, D_k] = -(\partial \psi^s / \partial t^k) D_s$. For details and explicit formulas see [1, 5, 32, 33]. As is well known, scalar differential invariants can be identified with functions on $J^\infty \pi$ invariant with respect to the fields \bar{U}_ψ and \bar{V}_a . These functions form a commutative associative \mathbb{R} -algebra, which can be thought of as algebra of functions on the orbit space $J^\infty \pi / (\mathfrak{G} \times \text{GL}_2)$.

We finish this section with a proposition giving the number N_r of independent r th order scalar differential invariants, as found by the classical method of Lie:

Proposition 2 *The dimension N_r of the orbit space $J^r \pi / (\mathfrak{G} \times \text{GL}_2)$ is given by the following table:*

r	0	1	2	3	4	...
N_r	0	4	14	28	46	...

The proof reduces to routine counting the number of independent equations in the system $\bar{U}_\psi f = 0$, $\bar{V}_a f = 0$ on $J^r \pi$. Computations are considerably simpler when using the Weyl–Lewis–Papapetrou metric coefficients (see Sect. 3.4 below).

Remark 1 A comment on a possible source of misunderstanding is due. Proposition 2 refers to scalar differential invariants as functions on the jet space $J^\infty \pi$. If such a function, say F , is evaluated for a particular metric \mathbf{g} , then it becomes a function on \mathcal{S} , which we shall denote as $F|_{\mathbf{g}}$ (formally $F|_{\mathbf{g}} = F \circ j^\infty \sigma_{\mathbf{g}}$, where j^∞ denotes a jet prolongation of a section of the bundle π and $\sigma_{\mathbf{g}}$ is the section associated with \mathbf{g}). Analogous correspondences hold for other geometric objects such as forms and vector fields. Hence another interpretation of scalar differential invariants as functions on \mathcal{S} .

Both interpretations are natural and indispensable. For instance, the order of an invariant can only be seen in the context of jet spaces, while the most natural way to construct an invariant consists in combining various invariant geometric constructions on \mathcal{S} , as demonstrated in the next section. It is usually harmless to use one and the same notation with both interpretations and omit the symbol $|_{\mathbf{g}}$. However, one should bear in mind that independence of functions on \mathcal{S} is very different from that on $J^\infty \pi$. The maximal number of independent functions is two on \mathcal{S} , and unlimited on $J^\infty \pi$.

3 The equivalence problem

3.1 First-order invariants

It is the aim of this section to provide explicit formulas for the four independent scalar invariants of the first order predicted in Proposition 2. We utilize invariant geometric objects associated with the metric g and with the triple of functions $h_{11}, h_{12} = h_{21}, h_{22}$ on \mathcal{S} . Indices are raised and lowered with the metric g . Comma notation is used for partial derivatives taken with respect to coordinates t^1, t^2 on \mathcal{S} .

Geometric objects on \mathcal{S} (functions, metrics) are obviously \mathfrak{G} -invariant. Therefore, of interest are GL_2 -invariant geometric objects on \mathcal{S} . Of them, GL_2 -invariant functions are the scalar invariants sought.

The obviously GL_2 -invariant metric g alone has no invariants of order less than two (it has one invariant of order 2, the scalar curvature R , and $k - 1$ more independent invariants of order k for each $k > 2$; see Żorawski [41]).

However, if $\alpha = \alpha_{ij} dt^i dt^j$ is another GL_2 -invariant quadratic (meaning bilinear symmetric) form on \mathcal{S} , then the trace $C_\alpha = \alpha_i^i = g^{ij} \alpha_{ij}$ is a GL_2 -invariant scalar. Moreover, the volume forms $\mathrm{vol}_g = \sqrt{|\det g|} dt^1 \wedge dt^2$ and $\mathrm{vol}_\alpha = \sqrt{|\det \alpha|} dt^1 \wedge dt^2$ are a multiple of each other, hence $Q_\alpha = \det \alpha / \det g$ is one more GL_2 -invariant scalar on \mathcal{S} . Alternatively, invariants C_α and Q_α can be defined as coefficients of the characteristic polynomial $\det(\alpha - \lambda g) / \det g$ in λ .

Denote by x the function

$$x = \det h$$

on \mathcal{S} . We already know that $x \neq 0$ everywhere since otherwise \mathbf{g} would be degenerate at some point. It is easily checked that x is invariant with respect to the subgroup $\mathrm{SL}_2 \subset \mathrm{GL}_2$, but not to its complement $\mathbb{R} \subset \mathrm{GL}_2$ consisting of scalings $x \mapsto cx$. The covector $X = dx/x$ is then invariant with respect to the full GL_2 since scalings

cancel out. Let ρ denote the degenerate quadratic form $(dx/x)^2 = (1/x^2)x_{,i}x_{,j} dt^i dt^j$ on \mathcal{S} . The trace

$$C_\rho = \frac{1}{x^2} g^{ij} x_{,i} x_{,j}$$

is a GL_2 -invariant function on \mathcal{S} , while Q_ρ is obviously zero.

Another GL_2 -invariant metric on \mathcal{S} is $\chi = (dh_{11} dh_{22} - dh_{12} dh_{12})/x$ with components

$$\chi_{ij} = \frac{1}{2x} \left(\begin{vmatrix} h_{11,i} & h_{12,j} \\ h_{21,i} & h_{22,j} \end{vmatrix} + \begin{vmatrix} h_{11,j} & h_{12,i} \\ h_{21,j} & h_{22,i} \end{vmatrix} \right).$$

Hence two more GL_2 -invariant functions on \mathcal{S} , namely

$$C_\chi = \frac{1}{x} g^{ij} \begin{vmatrix} h_{11,i} & h_{12,j} \\ h_{21,i} & h_{22,j} \end{vmatrix}, \quad Q_\chi = \frac{\det \chi}{\det g}.$$

Next consider linear combinations $\chi + c\rho$, which are GL_2 -invariant quadratic forms on \mathcal{S} as well. For the particular value $c = -\frac{1}{4}$ we obtain the *Cosgrove form*

$$\gamma = \chi - \frac{1}{4} \rho.$$

This name reflect the fact that γ coincides with [12, Eq. (2.3)]. While $C_\gamma = C_\chi - \frac{1}{4} C_\rho$, the other invariant

$$Q_\gamma = \frac{\det \gamma}{\det g} = \frac{1}{4x^3 \det g} \begin{vmatrix} h_{11} & h_{12} & h_{22} \\ h_{11,1} & h_{12,1} & h_{22,1} \\ h_{11,2} & h_{12,2} & h_{22,2} \end{vmatrix}^2 \quad (4)$$

is, in general, functionally independent of the invariants Q_χ , C_χ , C_ρ (on $J^1\pi$; see Remark 1). If so, then we choose C_ρ , C_χ , Q_χ , Q_γ to be the *basic first-order invariants* all other first-order scalar invariants are functions of.

3.2 Invariant frame

The invariant covector dx/x on \mathcal{S} can be turned into an invariant vector by rising its index. If done with the help of the metric g , the resulting vector X has components

$$X^i = g^{ij} \frac{1}{x} \frac{\partial x}{\partial t^j}.$$

If done with the help of the volume form vol_g , the resulting vector Y has components

$$Y^i = \epsilon^{ij} \frac{1}{x \sqrt{|\det g|}} \frac{\partial x}{\partial t^j},$$

where $\epsilon^{11} = \epsilon^{22} = 0$, $\epsilon^{12} = -\epsilon^{21} = -1$. It is perhaps worth noticing that $X \ln x = C_\rho$ while $Y \ln x = 0$.

Since

$$\begin{vmatrix} X^1 & Y^1 \\ X^2 & Y^2 \end{vmatrix} = \frac{C_\rho}{\sqrt{|\det g|}},$$

vectors X, Y form an invariant frame at each point of \mathcal{S} where $C_\rho \neq 0$, i.e., where X is not null. Components of g with respect to this frame are

$$g(X, X) = C_\rho, \quad g(X, Y) = 0, \quad g(Y, Y) = \pm C_\rho,$$

where the sign is that of $\det g$. Components of ρ are simply

$$\rho(X, X) = C_\rho^2, \quad \rho(X, Y) = 0, \quad \rho(Y, Y) = 0.$$

For components of χ we easily get

$$\begin{aligned} \chi(X, X) &= C_\rho C_\chi - 4Q_\chi + 4Q_\gamma, \\ \chi(Y, Y) &= \pm 4(Q_\chi - Q_\gamma), \\ \chi(X, X)\chi(Y, Y) - \chi(X, Y)^2 &= \pm C_\rho^2 Q_\chi, \end{aligned}$$

hence

$$\chi(X, Y) = \sqrt{\pm C_\rho^2 Q_\chi \mp 4(Q_\chi - Q_\gamma)(C_\rho C_\chi - 4Q_\chi + 4Q_\gamma)}.$$

3.3 Second-order scalar invariants

Lie derivatives of invariant objects with respect to invariant vector fields such as X, Y are invariant objects again. Moreover, the vector fields on $J^\infty \pi$ corresponding to X, Y (see Remark 1) are *invariant differentiations*, since they commute with the vector fields \bar{U}_ψ, \bar{V}_a for all ψ and a . In particular, if p is a first-order scalar invariant, then Xp, Yp are second-order scalar invariants.

Starting with the four basic first-order invariants $C_\rho, C_\chi, Q_\chi, Q_\gamma$, we obtain up to eight independent second-order invariants in this way. As we shall demonstrate below, these suffice for solving the classification problem in generic case.

Perhaps the best-known scalar second-order invariant of g is the scalar curvature R , which is the only independent invariant of second order (in terms of metric coefficients) on a two-dimensional manifold like \mathcal{S} .

The invariant vector fields X, Y are orthogonal with respect to g but do not commute in general. The coefficients a, b in the commutation relation $[X, Y] = aX + bY$ are second-order scalar differential invariants again.

Possible ways to construct invariants of second order include finding GL_2 -invariant quadratic forms on \mathcal{S} , such as the Lie derivatives $\mathfrak{L}_X \chi$ and $\mathfrak{L}_Y \chi$, or symmetric products $dp \odot dq$ for various choices of first-order scalar invariants p, q . Another obvious candidate is the Ricci tensor Ric_{ij} (which, unlike the Ricci tensor \mathbf{Ric}_{ij} of \mathbf{g} , does not necessarily vanish in case of vacuum metrics). Next, associated with any smooth function f on \mathcal{S} is the hessian $\text{Hess } f$, which is a quadratic form defined by $\text{Hess } f(Z, Z') = i_Z \nabla_{Z'} df$, i.e.,

$$\text{Hess}_{ij} f = f_{,ij} - \Gamma_{ij}^k f_{,k}.$$

Here ∇ is the Levi-Civita connection associated with g and Γ_{ij}^k are the usual Christoffel symbols for ∇ . The trace of $\text{Hess } f$ coincides with the Laplace–Beltrami operator

$$\Delta f = g^{ij} \text{Hess}_{ij} f.$$

Note that the order of $\text{Hess } f$ equals two plus the order of f . To obtain a GL_2 -invariant of second order, f must be of order zero. Choosing f to be the logarithm of the SL_2 -invariant x of order zero, we get

$$\nu_{ij} = \text{Hess}_{ij} \ln x = \frac{1}{x} \text{Hess}_{ij} x + \rho_{ij}$$

(note that scalings of x cancel out).

Therefore, a number of second-order invariants result as C_α, Q_α for α running through various linear combinations of $\text{Ric}_{ij}, \nu_{ij}, \rho_{ij}, \chi_{ij}, (\mathfrak{L}_X \chi)_{ij}, (\mathfrak{L}_Y \chi)_{ij}$.

On \mathcal{M} , we have of course also the geometric objects associated with the full metric \mathbf{g} . They can be restricted to \mathcal{S} , thereby providing another source of invariant objects. Since at most 14 second-order invariants can be independent (as functions on $J^2\pi$), it should not come as a surprise that, for instance, the restriction of the full Ricci tensor \mathbf{Ric}_{ij} is a linear combination of the Ricci tensor with ν, ρ and χ :

$$(\mathbf{Ric}|_{\mathcal{S}})_{ij} = \text{Ric}_{ij} - \frac{1}{2} \nu_{ij} - \frac{1}{4} \rho_{ij} + \frac{1}{2} \chi_{ij}, \quad i, j = 1, \dots, 2. \quad (5)$$

The remaining components of the full Ricci tensor can be written as

$$\mathbf{Ric}_{2+i, 2+j} = -\frac{1}{2} \Delta h_{ij} + \frac{1}{4} X h_{ij} - \frac{1}{2} C_\chi h_{ij} \quad i, j = 1, \dots, 2.$$

The full scalar curvatures are then related by

$$\mathbf{R} = R - C_\nu - \frac{1}{2} C_\rho + \frac{1}{2} C_\chi \quad (6)$$

with $C_\nu = \Delta \ln x$.

3.4 Weyl–Lewis–Papapetrou parametrization

The four basic invariants $C_\rho, C_\chi, Q_\chi, Q_\gamma$ look considerably simpler when the coefficients h_{ij} are expressed via the (three of four) Weyl–Lewis–Papapetrou [29] variables r, s, w (r, s nonzero) according to the identification

$$h = \frac{r}{s} [(du^1 + w du^2)^2 \mp s^2 (du^2)^2],$$

i.e.,

$$h = \begin{pmatrix} \frac{r}{s} & \frac{rw}{s} \\ \frac{rw}{s} & \frac{r(w^2 \mp s^2)}{s} \end{pmatrix} \quad (7)$$

in matrix notation. The minus sign corresponds to $\det h = -r^2 < 0$ and the plus sign to $\det h = r^2 > 0$. If ξ_1 is non-null, which can always be achieved, then $h_{11} \neq 0$ and $r = \sqrt{|\det h|}$, $s = r/h_{11}$, $w = h_{12}/h_{11}$.

Under this parametrization we have

$$x = \mp r^2, \quad \rho = \frac{4}{r^2} dr^2, \quad \gamma = \frac{1}{s^2} (ds^2 \mp dw^2). \quad (8)$$

Hence Cosgrove's [12] crucial observation that γ is a metric of constant curvature -1 if nondegenerate, i.e., if $Q_\gamma \neq 0$. Formulas

$$C_\gamma = C_\chi - \frac{1}{4} C_\rho = -g^{ij} \frac{s_{,i} s_{,j} \pm w_{,i} w_{,j}}{s^2}, \quad Q_\gamma = \mp \frac{1}{s^4 \det g} \begin{vmatrix} s_{,1} & w_{,1} \\ s_{,2} & w_{,2} \end{vmatrix}^2 \quad (9)$$

will be needed in the sequel. In case of Lorentz metrics we have $\det g \det h < 0$, hence $Q_\gamma \leq 0$.

Proposition 3 *If $Q_\gamma \neq 0$, then the Killing algebra of γ coincides with the \mathfrak{sl}_2 component of the algebra \mathfrak{gl}_2 of infinitesimal transformations (3) acting on Ξ .*

Proof. Both algebras have one and the same set of generators

$$\frac{\partial}{\partial w}, \quad s \frac{\partial}{\partial s} + w \frac{\partial}{\partial w}, \quad sw \frac{\partial}{\partial s} + \frac{1}{2} (w^2 \mp s^2) \frac{\partial}{\partial w}$$

in terms of Weyl–Lewis–Papapetrou variables. \square

Further simplification is achieved if taking into account that all two-dimensional metrics are conformally flat. Then g can be cast in one of the following two forms:

$$g = f(dt^1 dt^1 + dt^2 dt^2) \quad \text{or} \quad g = 2f dt^1 dt^2. \quad (10)$$

The full set of Weyl–Lewis–Papapetrou variables contains f as the fourth variable.

It should be noted that when fixing the explicitly conformally flat form of the metrics (10), we thereby impose further restriction on the pseudogroup \mathfrak{G} , namely, that the local diffeomorphisms be conformal maps. Fortunately, the classification problem of general two-dimensional metrics with respect to diffeomorphisms and the classification problem of metrics (10) with respect to conformal maps are identical. Using the Weyl–Lewis–Papapetrou coordinates r, s, w, f substantially simplifies computations on $J^\infty \pi$.

3.5 A solution to the equivalence problem

In this section we show how to reconstruct a generic metric \mathbf{g} up to isometry from the knowledge of functional dependences between certain eight invariants computed for \mathbf{g} . This provides a solution to the local equivalence problem as a particular case of the “principle of n invariants” [1, Ch. 7, § 4.3]. Let p, q be two independent scalar invariants, which p, q remain functionally independent when evaluated on a pseudo-Riemannian space \mathbf{g} (genericity). Then the metric \mathbf{g} is locally uniquely determined by the values of six components $\mathbf{g}_{ij}(p, q)$ with respect to coordinates p, q ; these components are themselves scalar invariants. Now, two metrics \mathbf{g}, \mathbf{g}' are locally equivalent if and only if the corresponding invariants $\mathbf{g}_{ij}(p, q), \mathbf{g}'_{ij}(p', q')$ have equal expressions in terms of invariants p, q and p', q' , respectively. That expressions for $\mathbf{g}_{ij}(p, q)$ can be large is not a problem, since it is p, q which must be simple enough for the procedure to be computationally tractable. In practice one usually finds a simpler set of invariants than $\mathbf{g}_{ij}(p, q)$, which is then suitable for storing in a database. Let \mathbf{g} be fixed in the sequel. In what follows we interpret invariants as objects on \mathcal{S} (cf. Remark 1).

Proposition 4 *Let p, q be two functionally independent invariants from the list $C_\rho, C_\chi, Q_\chi, Q_\gamma$ (on \mathcal{S}). Then knowing the values of the remaining two invariants and the invariants Xp, Yp, Xq, Yq in terms of p, q is sufficient for recovering the metric $\mathbf{g} \in \mathfrak{T}$.*

Recall that Xp, Yp, Xq, Yq are just four of the numerous second-order invariants of \mathbf{g} discussed in Section 3.3.

Proof. Once we know the values Xp, Yp, Xq, Yq in terms of p, q , we know the invariant frame X, Y in terms of coordinates p, q :

$$X = Xp \frac{\partial}{\partial p} + Xq \frac{\partial}{\partial q}, \quad Y = Yp \frac{\partial}{\partial p} + Yq \frac{\partial}{\partial q}.$$

Knowing also the remaining two values of $C_\rho, C_\chi, Q_\chi, Q_\gamma$ in terms of coordinates p, q , we can apply formulas given at the end of Section 3.2 to compute g, ρ, χ , and hence

also γ , in terms of coordinates p, q . In particular, g is fully and uniquely restored in this way.

To restore the remaining three components $h_{ij} = \mathbf{g}(\xi_i, \xi_j)$, we start with the determinant x , which is determined from $g(X, \cdot) = dx/x$ up to a constant multiplier. In terms of Weyl–Lewis–Papapetrou parameters r, w, s , we have $x = \mp r^2$, hence r is also restored up to a constant multiplier. To obtain w, s , we distinguish two cases.

If $Q_\gamma \neq 0$, then γ is the Cosgrove metric of constant curvature, i.e., a space form. As is well known, space forms are uniquely determined by their dimension and signature (see. e.g., Eisenhart [16, Ch. I, § 10]). Therefore, γ is isometric to the form $(1/s^2)(ds^2 \mp dw^2)$, meaning that w, s can be restored up to isometry of γ (isometries of γ reflect the freedom of determination of w and s by Proposition 3). This last step has been elaborated by Cosgrove [12] and reduced to solution of Appel equations.

If $Q_\gamma = 0$, then by (9) we have

$$\begin{vmatrix} s_{,1} & w_{,1} \\ s_{,2} & w_{,2} \end{vmatrix} = 0.$$

Therefore w, s are functionally dependent, i.e., $s = s(z), w = w(z)$ for some function z on \mathcal{S} , and

$$\gamma = \frac{s'^2 \mp w'^2}{s^2} (dz)^2$$

where prime denotes differentiation with respect to z . Assuming s non-constant, upon identification $z = s$ we have $\gamma = (1 \mp w_s^2)(ds/s)^2$, which determines $w(s)$ up to an additive constant. Assuming $s = \text{const}$ and $w \neq \text{const}$, the identification $z = w$ leads to $\gamma = \mp(1/s^2)(dw)^2$, which determines w up to an additive constant again. Assuming $s = \text{const}, w = \text{const}$ would lead to $Q_\gamma = Q_\chi = 0$ and $C_\chi = \frac{1}{4}C_\rho$, contradicting the assumptions of the proposition. \square

3.6 Relation to Ψ_2

The first-order invariants $C_\rho, C_\chi, Q_\chi, Q_\gamma$ are not necessarily invariant with respect to general diffeomorphisms, since by very construction they depend on the uniqueness property of the subalgebra \mathfrak{K} . In this section we relate Q_γ to $\text{Im } \Psi_2$ and C_χ to $\text{Re } \Psi_2$, where Ψ_2 is one of the Petrov invariants.

Consider the Weyl tensor $\mathbf{C}_{abcd} = \mathbf{Riem}_{abcd} - \mathbf{g}_{a[c}\mathbf{Ric}_{d]b} + \mathbf{g}_{b[c}\mathbf{Ric}_{d]a} + \frac{1}{3}\mathbf{R}\mathbf{g}_{a[c}\mathbf{g}_{d]b}$ in terms of adapted coordinates t^1, t^2, u^1, u^2 . Its nonzero components are

$$\begin{aligned} \mathbf{C}_{1234} &= \frac{1}{2} \sqrt{Q_\gamma \det h \det g}, \\ \mathbf{C}_{1212} &= \frac{1}{6} (R + \frac{1}{2} \Delta \ln x + \frac{1}{4} C_\rho - C_\chi) \det g, \\ \mathbf{C}_{3434} &= \frac{1}{6} (R + \frac{1}{2} \Delta \ln x + \frac{1}{4} C_\rho - C_\chi) \det h, \end{aligned}$$

components of the form \mathbf{C}_{ikjl} ($i, j = 1, 2$ and $k, l = 3, 4$), which are too involved to be presented here, and components obtained from these through the symmetries $\mathbf{C}_{abcd} = -\mathbf{C}_{bacd} = -\mathbf{C}_{abdc} = \mathbf{C}_{cdab}$. Due to its symmetries, the Weyl tensor can be thought of as acting on bivectors. The corresponding eigenvalue problem lies in the heart of the Petrov classification the Karlhede classification is a refinement of. To determine the Petrov type of a Lorentz metric, one examines the multiplicities of roots of the quartic $\Psi_0 x^4 - 4\Psi_1 x^3 y + 6\Psi_2 x^2 y^2 - 4\Psi_3 x y^3 + \Psi_4 y^4$, where $\Psi_0 = \mathbf{C}_{abcd} k^a m^b k^c m^d$, $\Psi_1 =$

$C_{abcd}k^ak^bl^cm^d$, $\Psi_2 = C_{abcd}k^am^b\bar{m}^cl^d$, $\Psi_3 = C_{abcd}k^al^b\bar{m}^cm^d$, $\Psi_4 = C_{abcd}\bar{m}^al^bk\bar{m}^cl^d$ are calculated with respect to an arbitrary complex null tetrad (k, l, m, \bar{m}) . Routine computation shows that $\Psi_1 = 0$ and $\Psi_3 = 0$ (hence metrics in \mathfrak{T} cannot be of Petrov type *III*) and then the quartic in question becomes $\Psi_0X^4 + 6\Psi_2X^2Y^2 + \Psi_4Y^4$. A metric is algebraically special if and only if $\Psi_0\Psi_4(9\Psi_2^2 - \Psi_0\Psi_4) = 0$. Moreover, $\Psi_0\Psi_4$ and Ψ_2 are invariants with respect to general diffeomorphisms. Of special interest in this paper is the invariant Ψ_2 and its real and imaginary part. By routine computation

$$\begin{aligned}\operatorname{Re} \Psi_2 &= \frac{1}{16} C_\chi - \frac{1}{8} R + \frac{1}{24} \mathbf{R} \\ \operatorname{Im} \Psi_2 &= \frac{1}{4} \sqrt{-Q_\gamma}\end{aligned}\tag{11}$$

(recall that $Q_\gamma \leq 0$ in case of Lorentz metrics). Thus, $\operatorname{Im} \Psi_2$ is of the first order.

3.7 Genericity

It is the aim of this section to review the genericity assumptions we made in the course of exposition and express them in terms of scalar invariants. Here is the full list of the assumptions (additionally to $\det g \neq 0$ and $x = \det h \neq 0$, which follow from the nondegeneracy of \mathbf{g}):

1. The algebra of Killing vectors has a unique two-dimensional commutative sub-algebra \mathfrak{K} (Section 2);
2. $C_\rho \neq 0$ (Section 3.2);
3. There are at least two functionally independent invariants among C_ρ , C_χ , Q_χ , Q_γ (Proposition 4).

The first of these assumptions, unlike the others, is not formulated in terms of invariants. Moreover, it cannot be formulated in terms of invariants that themselves depend on the uniqueness property of \mathfrak{K} . Yet we can improve the situation with the help of a classical result valid without restriction to the class \mathfrak{T} .

Proposition 5 *Let \mathbf{g} be a general metric (possibly outside class \mathfrak{T}), for which there exist two functionally independent scalar invariants with respect to the general diffeomorphisms. Then the distribution generated by Killing vectors of \mathbf{g} is of dimension at most two.*

Proof. More general results are proved in Kerr [25, 26]. A simple proof goes as follows. Assume that ξ_1, ξ_2, ξ_3 are three Killing vectors of \mathbf{g} such that the distribution Ξ they generate is of dimension three. Let p, q be scalar differential invariants. Then dp vanishes on Ξ , since $0 = \xi_i p = dp(\xi_i)$, and similarly dq vanishes on Ξ . But \mathcal{M} has only one dimension more than that of Ξ , whence $dp \wedge dq = 0$, proving that invariants p, q are functionally dependent. \square

Turning back to metrics of class \mathfrak{T} , our Q_γ is invariant with respect to general diffeomorphisms in view of results of the preceding section. The same is true about C_χ in case of vacuum Einstein metrics.

Now we are left with the question when there exists a third Killing vector lying in the distribution Ξ generated by \mathfrak{K} .

Proposition 6 *The algebra of Killing vectors spanning the distribution Ξ is three-dimensional if and only if coefficients h_{ij} are constant multiples of each other and two-dimensional otherwise.*

Proof. In the adapted coordinates we have $\xi_1 = \partial/\partial u^1$, $\xi_2 = \partial/\partial u^2$, $\xi_3 = f^1 \partial/\partial u^1 + f^2 \partial/\partial u^2$ for some functions f^i of t^1, t^2, u^1, u^2 . The Killing equations $\xi_3 \mathbf{g} = 0$ reduce to

$$h_{ik} \frac{\partial f^k}{\partial t^j} = 0, \quad i, j = 1, 2, \quad (12)$$

$$h_{ik} \frac{\partial f^k}{\partial u^j} + h_{kj} \frac{\partial f^k}{\partial u^i} = 0, \quad i, j = 1, 2. \quad (13)$$

Since $\det h \neq 0$, equations (12) imply that f^i do not depend on t^1, t^2 .

Functions $f^1(u^1, u^2)$, $f^2(u^1, u^2)$ now satisfy the remaining equations (13) and also derivatives of equations (13) with respect to t^1, t^2 . The resulting nine homogeneous linear equations can be viewed as an algebraic system in four unknowns $\partial f^i / \partial u^k$, $i, k = 1, 2$. Its properties depend on the values of six 4×4 minors D_j , $j = 1, \dots, 6$, formed by the rows corresponding to the three equations (13) plus one of the six derivatives thereof. These minors are

$$\begin{aligned} D_1 &= 8(h_{11}h_{12,1} - h_{12}h_{11,1}) \det h, & D_4 &= 8(h_{11}h_{12,2} - h_{12}h_{11,2}) \det h, \\ D_2 &= 4(h_{11}h_{22,1} - h_{22}h_{11,1}) \det h, & D_5 &= 4(h_{11}h_{22,2} - h_{22}h_{11,2}) \det h, \\ D_3 &= 8(h_{12}h_{22,1} - h_{22}h_{12,1}) \det h, & D_6 &= 8(h_{12}h_{22,2} - h_{22}h_{12,2}) \det h. \end{aligned}$$

If at least one of them is nonzero, then the algebraic system (13) has only the zero solution, hence f^1, f^2 are constants and ξ_3 belongs to \mathfrak{K} .

If, conversely, $D_j = 0$ for all $j = 1, \dots, 6$, then

$$h_{ij} = H c_{ij} \quad (14)$$

where $H(t^1, t^2)$ is a nonzero scalar function and c_{ij} is a nonsingular symmetric constant matrix (meaning that coefficients h_{ij} are constant multiples of each other). Under (14), system (13) has a unique solution

$$\xi_3 = -(c_{11}u^1 + c_{12}u^2) \frac{\partial}{\partial u^2} + (c_{12}u^1 + c_{22}u^2) \frac{\partial}{\partial u^1}$$

modulo \mathfrak{K} and up to a constant multiple. \square

By Proposition 6 we have either $\xi_3 \in \mathfrak{K}$ or (14). In the latter case $C_\chi = 4C_\rho$, $Q_\chi = 0$, $Q_\gamma = 0$, contradicting assumption 3.

3.8 Vacuum case

We consider Einstein's vacuum equations with cosmological constant

$$\mathbf{Ric}_{ij} - \frac{1}{2} \mathbf{R} \mathbf{g}_{ij} + \Lambda \mathbf{g}_{ij} = 0 \quad (15)$$

where Λ denotes the cosmological constant.

Proposition 7 *Let $C_\gamma^2 \neq 4Q_\gamma$. Let $p = C_\rho$, let q be an arbitrary invariant from the list C_χ, Q_χ, Q_γ , such that p, q are functionally independent (on \mathcal{S}). Then knowing the values of the remaining two invariants in terms of p, q are sufficient for recovering a generic metric $\mathbf{g} \in \mathfrak{T}$ in the vacuum case.*

Proof. Using the Weyl–Lewis–Papapetrou parametrization, we consider the system consisting of six Einstein’s vacuum equations with cosmological constant (15) and eight equations for $XC_\rho, XC_\chi, XQ_\chi, XQ_\gamma, YC_\rho, YC_\chi, YQ_\chi, YQ_\gamma$. This system of 14 equations is linear in second derivatives of the four functions f, s, r, w . The number of second derivatives is ten (f_{tt} and f_{zz} are missing). By applying linear algebra we obtain four nontrivial linear dependences among the second order invariants and components of Einstein’s vacuum equations with cosmological constant. Rewritten in invariant form, they are

$$\begin{aligned} YC_\rho &= -8I_1, \\ XC_\rho &= 8(Q_\gamma - Q_\chi) + C_\rho(C_\gamma - \tfrac{3}{4}C_\rho - 4L), \\ \tfrac{1}{4}C_\gamma I_1 XC_\chi - \tfrac{1}{2}I_1 XQ_\gamma + \tfrac{1}{2}I_2 YQ_\gamma - \tfrac{1}{2}I_3 YC_\chi - I_4 YQ_\chi \\ &= I_1[I_4(C_\gamma - \tfrac{1}{2}C_\rho + 4\Lambda) - \tfrac{1}{16}C_\gamma C_\rho(\tfrac{3}{4}C_\rho + 4\Lambda)], \\ \tfrac{1}{4}\varepsilon C_\gamma I_1 YC_\chi - \tfrac{1}{2}\varepsilon I_1 YQ_\gamma - \tfrac{1}{2}I_2 XQ_\gamma + \tfrac{1}{2}I_3 XC_\chi + I_4 XQ_\chi \\ &= I_4((\tfrac{1}{2}C_\rho + 4\Lambda)(Q_\gamma - Q_\chi) + 2I_3) - \tfrac{1}{8}I_3 C_\rho(\tfrac{3}{4}C_\rho + 4\Lambda) \end{aligned}$$

where we denoted

$$\begin{aligned} I_1^2 &= \varepsilon[(Q_\gamma - Q_\chi)^2 - \tfrac{1}{4}C_\rho(C_\gamma Q_\chi - (C_\gamma + \tfrac{1}{4}C_\rho)Q_\gamma)], \\ I_2 &= -(Q_\gamma + Q_\chi) + \tfrac{1}{2}C_\gamma(C_\gamma + \tfrac{1}{4}C_\rho), \\ I_3 &= \tfrac{1}{2}C_\gamma(Q_\gamma - Q_\chi) + \tfrac{1}{4}C_\rho Q_\gamma, \\ I_4 &= \tfrac{1}{4}C_\gamma^2 - Q_\gamma, \\ C_\rho &= 4(C_\chi - C_\gamma), \\ \varepsilon &= \pm 1 \end{aligned}$$

(minus in elliptic case and plus in hyperbolic case).

The derivation of these relations is tedious but straightforward and relies on inversion of a matrix with determinant $C_\rho I_4$. While $C_\rho \neq 0$ is nonzero by assumptions (Section 3.7), $I_4 \neq 0$ is an additional genericity assumption.

To finish the proof, we express Xp, Yp , where $p = C_\rho$, from the first and the second equation and Xq, Yq , where q is one of C_χ, Q_χ, Q_γ , from the third and the fourth equation. By Proposition 4 this is sufficient for recovering the metric $\mathbf{g} \in \mathfrak{T}$. \square

For instance, if $q = Q_\chi$, then expressing Xq, Yq is possible if and only if $I_4 \neq 0$.

Proposition 8 *In the vacuum case, $\text{Re } \Psi_2$ is of the first order.*

Proof. On one hand the trace of $(\mathbf{Ric}|_S)_{ij}$ is

$$g^{ij}(\mathbf{Ric}|_S)_{ij} = \mathbf{R} - 2\Lambda \tag{16}$$

by (15). On the other hand the trace of (5) is

$$g^{ij}(\mathbf{Ric}|_S)_{ij} = R - \tfrac{1}{2}C_\nu - \tfrac{1}{4}C_\rho + \tfrac{1}{2}C_\chi. \tag{17}$$

From (16), (17) and by (6) we have

$$C_\nu = -\tfrac{1}{2}C_\rho - 4\Lambda. \tag{18}$$

Thus, C_ν is a first order invariant in the vacuum case. The trace of (15) is

$$\mathbf{R} = 4\Lambda. \quad (19)$$

Therefore, in the vacuum case we have

$$R = -\frac{1}{2} C_\chi. \quad (20)$$

by (16), (17), (18) and (19). Then

$$\text{Re } \Psi_2 = \frac{1}{8} C_\chi + \frac{1}{6} \Lambda$$

by (11). \square

The other Petrov invariant $\Psi_0\Psi_4$ is of second order even in vacuum.

4 Example

Example 1 We illustrate our results by classifying the Kerr–NUT–(anti)de Sitter space-time, attributed to Plebański and Demiański [37, 39], who derived it a special case of a seven-parameter family of Petrov type D solutions with nonzero cosmological constant Λ . Denoting the nonignorable coordinates as t_1, t_2 , the metric coefficients are

$$h_{11} = \frac{P}{t_1^2 + t_2^2} - \frac{Q}{t_1^2 + t_2^2}, \quad h_{12} = \frac{Pt_2^2}{t_1^2 + t_2^2} + \frac{Qt_1^2}{t_1^2 + t_2^2}, \quad h_{22} = \frac{Pt_2^4}{t_1^2 + t_2^2} - \frac{Qt_1^4}{t_1^2 + t_2^2},$$

and

$$g_{11} = \frac{t_1^2 + t_2^2}{P}, \quad g_{12} = 0, \quad g_{22} = \frac{t_1^2 + t_2^2}{Q}$$

where

$$P = (A^2 - t_1^2)(1 + \frac{1}{3} \Lambda t_1^2) + 2Lt_1, \quad Q = (A^2 + t_2^2)(1 - \frac{1}{3} \Lambda t_2^2) - 2Mt_2,$$

M, L, A, Λ being parameters of the metric. While M is the mass of the source, the physical meaning of L, A is not immediately obvious. Griffiths and Podolský [18] gave the following identification

$$A^2 = (a^2 - l^2) \frac{(a^2 + 3l^2)\Lambda + 3}{(a^2 - l^2)\Lambda + 3},$$

$$L = l \left[\frac{1}{3} (a^2 - l^2)\Lambda + \frac{(a^2 + 3l^2)\Lambda + 3}{(a^2 - l^2)\Lambda + 3} \right],$$

where a is the angular momentum and l is the NUT parameter.

The four basic first-order invariants $C_\rho, C_\chi, Q_\chi, Q_\gamma$ can be routinely computed. Two of them turn out to be particularly simple, hence we choose them as the independent invariants p, q :

$$p = Q_\gamma = -4 \left[\frac{Lt_2(t_2^2 - 3t_1^2) - Mt_1(3t_2^2 - t_1^2)}{(t_1^2 + t_2^2)^3} \right]^2, \quad (21)$$

$$q = C_\chi = 4 \frac{Mt_2(t_2^2 - 3t_1^2) + Lt_1(3t_2^2 - t_1^2)}{(t_1^2 + t_2^2)^3} - \frac{4}{3} \Lambda.$$

Components of the invariant frame are easily seen to be

$$X^1 = -\frac{2}{3} \frac{2\Lambda t_1^3 - (\Lambda A^2 - 3)t_1 - 3L}{t_1^2 + t_2^2},$$

$$X^2 = -\frac{2}{3} \frac{2\Lambda t_2^3 + (\Lambda A^2 - 3)t_2 + 3M}{t_1^2 + t_2^2},$$

$$Y^1 = \operatorname{sgn} Q \sqrt{\left| \frac{P}{Q} \right|} X^2,$$

$$Y^2 = -\operatorname{sgn} P \sqrt{\left| \frac{Q}{P} \right|} X^1.$$

Expressions for the other invariants needed for recognizing equivalence are quite large. They are schematically given in the following table:

Invariant	Numerator	Denominator
C_ρ	$p_{10,6,6}$	$(t_1^2 + t_2^2)PQ$
Q_χ	$p_{17,14,14}$	$(t_1^2 + t_2^2)^6 PQ$
XQ_γ	$p_{3,3,3} p_{7,7,7}$	$(t_1^2 + t_2^2)^8$
XC_χ	$p_{7,7,7}$	$(t_1^2 + t_2^2)^5$
YQ_γ	$p_{3,3,3} p_{11,8,8}$	$(t_1^2 + t_2^2)^8 P^{1/2} Q^{1/2}$
YC_χ	$p_{11,8,8}$	$(t_1^2 + t_2^2)^5 P^{1/2} Q^{1/2}$

Here p_{n,n_1,n_2} is a substitute for a polynomial $p(t_1, t_2)$ of total degree n , degree n_1 in the indeterminate t_1 and degree n_2 in the indeterminate t_2 (e.g., the substitute for P would be $p_{4,4,0}$).

Actual values of these invariants are suitable for storing in a database of exact solutions. Prior to that one can wish to express all of them in terms of p, q . This usually constitutes the main technical difficulty connected with equivalence analysis. In our example a help comes from the simple relation

$$\left(\frac{4}{3}\Lambda + C_\chi\right)^2 - 4Q_\gamma = 16 \frac{M^2 + L^2}{(t_1^2 + t_2^2)^3}.$$

Inserted back to the denominators in (21), this allows us to find the following cubic equations for t_1 and t_2 in terms of invariants p, q and parameters Λ, M, L :

$$4t_1^3 - 3It_1 + I_+ = 0,$$

$$4t_2^3 - 3It_2 - I_- = 0,$$

where

$$I = \left[16 \frac{M^2 + L^2}{(C_\chi + \frac{4}{3}\Lambda)^2 - 4Q_\gamma} \right]^{1/3}, \quad I_\pm = 4 \frac{M(C_\chi + \frac{4}{3}\Lambda) \pm 2L\sqrt{-Q_\gamma}}{(C_\chi + \frac{4}{3}\Lambda)^2 - 4Q_\gamma}.$$

One easily checks that $I_4 \neq 0$ so that by Proposition 7 the metric is characterizable by expressions for C_χ, Q_γ in terms of C_ρ, Q_χ .

Example 2 In the special case of $\Lambda = 0$ (Kerr–NUT solution) we have $L = l$, $P = -t_1^2 + A^2 + 2Lt_1$, $Q = t_2^2 + A^2 - 2Mt_2$ quadratic in t_1, t_2 , respectively, and the invariants simplify considerably:

Invariant	Numerator	Denominator
C_ρ	$p_{2,2,2}$	$(t_1^2 + t_2^2)PQ$
Q_χ	$p_{10,8,8}$	$(t_1^2 + t_2^2)^6 PQ$
XQ_γ	$p_{3,3,3} p_{5,5,5}$	$(t_1^2 + t_2^2)^8$
XC_χ	$p_{5,5,5}$	$(t_1^2 + t_2^2)^5$
YQ_γ	$p_{3,3,3} p_{7,6,6}$	$(t_1^2 + t_2^2)^8 P^{1/2} Q^{1/2}$
YC_χ	$p_{7,6,6}$	$(t_1^2 + t_2^2)^5 P^{1/2} Q^{1/2}$

The same Kerr–NUT solution has been classified by Åman and Karlhede in [2]. These authors start with the metric [27, II.A], given in the Newman–Penrose tetrad formalism. Since this metric is not in the form (1), the results cannot be compared immediately. In the course of the Karlhede algorithm applied to this metric, 14 curvature invariants are computed. These are complex quantities, whose real and imaginary parts are to be considered separately. This example confirms that a tailored procedure can lead to a significant reduction of the total number of computed invariants.

On the other hand, all the Åman–Karlhede invariants already have (nearly) explicit expression in terms of two complex conjugated quantities p, p^* , which themselves are easily derivable from the unique complex curvature invariant of order zero. Thus the “last step” becomes rather trivial in this case. However, this may be attached to the particular way the Kerr–NUT metric (which is of Petrov type D) has been derived.

5 Conclusions

We have demonstrated that four first-order scalar differential invariants and two first-order invariant vector fields suffice to solve the local equivalence problem within the class \mathfrak{T} of pseudo-Riemannian metrics with two commuting orthogonally transitive vector fields. Results are restricted to generic metrics in \mathfrak{T} as identified in Sect. 3.7, leaving the special subclasses unexplored. The main open problems include: the description of the special (nongeneric) subclasses exempt from consideration in this paper; the classification problem within these classes; the comparison to the Karlhede algorithm; finding solutions to the Einstein equations in terms of first-order invariants (cf. [7] and references therein).

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